

Local distinguishability of any three quantum states

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2009 J. Phys. A: Math. Theor. 42 072002

(<http://iopscience.iop.org/1751-8121/42/7/072002>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.156

The article was downloaded on 03/06/2010 at 08:29

Please note that [terms and conditions apply](#).

FAST TRACK COMMUNICATION

Local distinguishability of any three quantum states

Somshubhro Bandyopadhyay^{1,2,4} and Jonathan Walgate^{1,3}¹ Institute for Quantum Information Science, University of Calgary, Alberta T2N 1N4, Canada² DIRO, Université de Montréal, C P 6128, Succursale Centre-Ville, Montréal, Québec H3C 3J7, Canada³ Perimeter Institute for Theoretical Physics, 31 Caroline St N, Waterloo, Ontario N2 L 2Y5, CanadaE-mail: bandyo@iro.umontreal.ca and jwalgate@perimeterinstitute.ca

Received 8 December 2008, in final form 7 January 2009

Published 21 January 2009

Online at stacks.iop.org/JPhysA/42/072002**Abstract**

We prove that any three linearly independent pure quantum states can always be locally distinguished with nonzero probability regardless of their dimension, entanglement or multipartite structure. Almost always, all three states can be unambiguously identified. The only exceptional case, where one state is locally knowable but the other two are not, is found among multi-qubit states.

PACS numbers: 89.70.+c, 03.65.-w

Global operations on a quantum system can process information in ways that local operations on the system's parts cannot. All uses of entanglement in quantum information theory flow from this one fact, from teleportation [1] to Shor's factoring algorithm [2]. However a fundamental question remains unanswered. When is global information about a quantum system also available locally? This question can be formally posed as a local state discrimination task. Given one copy of a system in one of a known set of quantum states $\{|\psi_i\rangle\}$, how much 'which state' information can be gleaned by local operations and classical communication (LOCC), and how much more information is revealed by global measurements?

This problem has attracted much attention in recent years, after surprising results showed perfect local distinguishability was not directly linked to entanglement. Bennett and coworkers presented sets of orthogonal unentangled states that were not perfectly locally distinguishable [3]. JW, Short, Hardy and Vedral proved orthogonal pairs of states are always perfectly locally distinguishable, irrespective of their entanglement [4].

There are two natural approaches to quantum state discrimination. Optimal discrimination seeks the best possible guess as to the state of the system [5]. Conclusive discrimination (also called unambiguous discrimination) seeks certain knowledge of the state of the system, balanced against a possibility of failure [6]. In particular, N linearly independent quantum

⁴ Present address: Department of Physics, Bose Institute, Kolkata 700009, India. E-mail: som@bosemain.boseinst.ac.in

states can be conclusively distinguished by a $(N + 1)$ -outcome generalized measurement, where N outcomes correspond to correct (that is, error free) identification of the quantum states, and the remaining outcome corresponds to an inconclusive result, for which one fails to correctly identify the unknown quantum state. Note that conclusive identification of every state is not possible if the states are linearly dependent [7].

It follows directly from the results of Walgate *et al* [4] and Virmani *et al* [8] that local parties can always gain some amount of ‘which state’ information about the pure state of a shared system, and use it to improve their guesswork. Optimal state discrimination is always locally feasible in this sense, although the local optimum may be significantly worse than the global. Conclusive discrimination is more interesting. All pairs of pure quantum states can be conclusively discriminated equally well locally and globally [9]. Generically, a small number of pure states (proportional to the dimension of the subsystems) can be conclusively discriminated with non-negligible probability [15]. But in every multipartite dimensionality there are sets of four pure quantum states that are not conclusively locally distinguishable *at all*. In this case local parties can never gain certain knowledge of which state they possess; the Bell states are the simplest example of such a set [10].

So two states are always conclusively distinguishable, and four states can be conclusively indistinguishable. We complete the picture by showing that provided they are linearly independent (only linearly independent states are globally distinguishable) three pure quantum states can be conclusively locally distinguished. Local protocols may not succeed as often as global measurements, but they can succeed some of the time. No triplet of pure states, no matter how entangled, conceals any fraction of its ‘which state’ information from local parties with certainty.

We present our results in the following framework. A multipartite quantum system Q is shared between n different local parties, each with access to one of n local Hilbert spaces: $\mathcal{H}_Q = \bigotimes_{j=1}^n \mathcal{H}_j$. It has been prepared in one of a known set of possible pure states $\mathcal{S} = \{|\psi_i\rangle\}$, each with some nonzero (but potentially unknown) probability p_i . The local parties are set the task of discovering with certainty which of the states \mathcal{S} they have been given, using only LOCC. We will use the following definitions.

Definition 1. A state $|\psi_i\rangle \in \mathcal{S}$ is *conclusively locally identifiable* if and only if there is a LOCC protocol whereby with some nonzero probability $p > 0$ it can be determined that Q was certainly prepared in state $|\psi_i\rangle$.

Definition 2. A set of states \mathcal{S} is *conclusively locally distinguishable* if and only if every state in \mathcal{S} is *conclusively locally identifiable*.

Conclusive state identification has qualitative links to entanglement. It was proved by Horodecki *et al* that the states of a complete orthonormal basis are conclusively locally identifiable states if and only if they are product states [11]. We show below in corollary 1 that if no members of an incomplete basis of orthogonal states are conclusively identifiable then the set must be completely entangled.

We begin by establishing a necessary and sufficient condition for a set of states to be conclusively locally distinguishable, first proved by Chefles [12]. We outline a simplified version of Chefles’ specific to the case of pure states. We will then show how this condition holds for sets of three states.

Lemma 1 (Chefles). Let a multipartite quantum system Q be prepared in one of a set of pure, linearly independent multipartite quantum states $\mathcal{S} = \{|\psi_i\rangle\}$. Let $|\psi_x\rangle \in \mathcal{S}$.

If and only if there exists a product state $|\phi\rangle$ such that $\forall i \neq x \langle \psi_i | \phi \rangle = 0$, and $\langle \psi_x | \phi \rangle \neq 0$, then $|\psi_x\rangle$ is *conclusively locally identifiable* in \mathcal{S} .

Proof of sufficiency. Assume a product's state $|\phi\rangle$ with the above properties exists. The parties can locally project into a product basis of \mathcal{H}_Q that includes $|\phi\rangle$. If the state of Q is $|\psi_x\rangle$ they will obtain the result projecting onto $|\phi\rangle$ with probability $|\langle\psi_x|\phi\rangle|^2$, which is greater than zero. In this case, they have conclusively locally identified $|\psi_x\rangle$ since no other state $|\psi_i\rangle$ ever yields this projection result. \square

Proof of necessity. Assume that $|\psi_x\rangle$ is conclusively locally identifiable. There is a LOCC protocol, describable by a separable superoperator, which can produce at least one measurement outcome conclusively identifying $|\psi_x\rangle$. This outcome corresponds to some separable POVM element $M^\dagger M = A^\dagger A \otimes B^\dagger B \otimes \dots$, which because it identifies $|\psi_x\rangle$ must satisfy $\forall i \neq x \langle\psi_i|M^\dagger M|\psi_i\rangle = 0$, and $\langle\psi_x|M^\dagger M|\psi_x\rangle \neq 0$. $M^\dagger M$ is decomposable into a set of rank-one projection operators onto product states $\{|P_l\rangle\}$: $M^\dagger M = \sum_{j,k\dots} A_j^\dagger A_j \otimes B_k^\dagger B_k \otimes \dots = \sum_l (|P_l\rangle\langle P_l|)^\dagger (|P_l\rangle\langle P_l|)$. These product states must satisfy $\forall l \forall i \neq x \langle\psi_i|P_l^\dagger P_l|\psi_i\rangle = 0$ and $\exists l \langle\psi_x|P_l^\dagger P_l|\psi_x\rangle \neq 0$. Let the product state satisfying both conditions be $|\phi\rangle$. Thus there exists a product state $|\phi\rangle$ such that $\forall i \neq x \langle\psi_i|\phi\rangle = 0$ and $\langle\psi_x|\phi\rangle \neq 0$. \square

Corollary 1. *All product states belonging to sets of pure orthogonal states are conclusively locally identifiable and the subset of unentangled members of such a set is conclusively locally distinguishable.*

Proof. If $\mathcal{S} = \{|\psi_i\rangle\}$ is a set of orthogonal pure states, and $|\psi_x\rangle \in \mathcal{S}$ is a product state, then $|\phi\rangle = |\psi_x\rangle$ satisfies the sufficient condition of lemma 1. \square

Interestingly, although sets of pure orthogonal states must be completely entangled in order to be completely conclusively indistinguishable, linearly independent states are not so restricted. In fact, Duan *et al* have recently shown that there are sets of product states that are completely conclusively indistinguishable—a strong form of ‘nonlocality without entanglement’ [14]. JW and Scott have shown that generic sets of states obey the same numerical threshold for conclusive distinguishability whether they are entangled or not [15].

Theorem 1. *Let a multipartite quantum system Q be prepared in one of a set of three pure, linearly independent multipartite quantum states $\mathcal{S} = \{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\}$.*

There exists $|\psi_x\rangle \in \mathcal{S}$ such $|\psi_x\rangle$ is conclusively locally identifiable.

We will prove this result separately for three different cases. First we will deal with systems whose three possible states cannot be composed on a chain of qubits (i.e. where $\text{Span}(\mathcal{S}) \subset \mathcal{H}_Q \neq \bigotimes_{i=1}^n \mathcal{H}_2$), with each local party holding just one qubit). Then we will consider $\mathcal{H}_2 \otimes \mathcal{H}_2$ systems. Lastly, we will prove our result for larger arrays of qubits: $\mathcal{H}_Q = \bigotimes_{i=1}^{n>2} \mathcal{H}_2$. These three cases cover all possible multipartite situations.

Lemma 2. (higher-dimensional states) *Let a multipartite quantum system Q be prepared in one of a set of three pure, linearly independent multipartite quantum states $\mathcal{S} = \{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\}$. Let the space spanned by \mathcal{S} be such that it cannot be expressed in the form $\bigotimes_{i=1}^n \mathcal{H}_2$.*

\mathcal{S} is conclusively locally distinguishable.

Proof. If the space spanned by \mathcal{S} cannot be expressed in the form $\bigotimes_{i=1}^n \mathcal{H}_2$, then at least one of the local parties has an irreducibly three- or higher-dimensional Hilbert space \mathcal{H}_i . We call this party ‘Alice’. We can thus write the states:

$$\begin{aligned}
 |\psi_1\rangle &= \sum_i a_i |i\rangle_A |\eta_i\rangle_{BC\dots}, \\
 |\psi_2\rangle &= \sum_i b_i |i\rangle_A |v_i\rangle_{BC\dots}, \\
 |\psi_3\rangle &= \sum_i c_i |i\rangle_A |\mu_i\rangle_{BC\dots},
 \end{aligned} \tag{1}$$

where the vectors $|\eta_i\rangle_{BC\dots}$, $|v_i\rangle_{BC\dots}$ and $|\mu_i\rangle_{BC\dots}$ are normalized, and a_i , b_i and c_i are complex coefficients satisfying $\sum_i a_i^* a_i = 1$. Following the strategy of lemma 1, we will show that there exists a product state $|\phi\rangle$ such that $\langle\psi_1|\phi\rangle = \langle\psi_2|\phi\rangle = 0$ and $\langle\psi_3|\phi\rangle \neq 0$. Let us thus write the product state:

$$|\phi\rangle = \left(\sum_i x_i |i\rangle_A \right) \otimes |\theta\rangle_{BC\dots}$$

with $\sum_i x_i^* x_i = 1$. We choose $|\theta\rangle$ such that it is a product state amongst the parties $B, C \dots$ and so that equations (2) are linearly independent. We can always do this. (Note in fact that a randomly chosen $|\theta\rangle$ will have this property with probability one, thanks to the linear independence of the states in \mathcal{S} .) $|\phi\rangle$ must satisfy the following conditions:

$$\begin{aligned}
 \langle\psi_1|\phi\rangle &= \sum_i x_i a_i^* \langle\eta_i|\theta\rangle = 0, \\
 \langle\psi_2|\phi\rangle &= \sum_i x_i b_i^* \langle v_i|\theta\rangle = 0, \\
 \langle\psi_3|\phi\rangle &= \sum_i x_i c_i^* \langle\mu_i|\theta\rangle \neq 0.
 \end{aligned} \tag{2}$$

The quantities $a_i^* \langle\eta_i|\theta\rangle$, $b_i^* \langle v_i|\theta\rangle$ and $c_i^* \langle\mu_i|\theta\rangle$ are all fixed by our arbitrary choice of basis $\{|i\rangle_A\}$, and product state $|\theta\rangle_{BC\dots}$. There are at least three variables x_i , because $\mathcal{H}_A \neq 2$. With three linearly independent equations and three variables, there is always a solution for the x_i . (Note that normalization does not further restrict the solution of these equations, as they only specify sums to ‘zero’ or ‘not zero’.) Therefore, we can always find a product state $|\phi\rangle$ that is orthogonal to $|\psi_1\rangle$ and $|\psi_2\rangle$, but nonorthogonal to $|\psi_3\rangle$. By lemma 1, this means $|\psi_3\rangle$ is conclusively locally identifiable in \mathcal{S} .

The same reasoning applies to $|\psi_1\rangle$ and $|\psi_2\rangle$, so \mathcal{S} is conclusively locally distinguishable. □

Surprisingly, the only exceptions to this ‘one for all and all for one’ structure are found amongst the simplest quantum systems—qubits.

Lemma 3 (two qubits). *Let a multipartite quantum system Q be prepared in one of a set of three pure, linearly independent multipartite quantum states $\mathcal{S} = \{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\}$. Let $\mathcal{H}_Q = \mathcal{H}_2 \otimes \mathcal{H}_2$.*

There exists $|\psi_x\rangle \in \mathcal{S}$ such $|\psi_x\rangle$ is conclusively locally identifiable.

Proof. Either at least two of the three members of \mathcal{S} are product states, or else at least two of them are entangled states. Whichever is the case, we label the states such that $|\psi_1\rangle$ and $|\psi_2\rangle$ are similar—either they are both product states, or they are both entangled. We will show that $|\psi_3\rangle$ is then conclusively locally identifiable. $|\psi_1\rangle$ and $|\psi_2\rangle$ are linearly independent and span a two-dimensional subspace of \mathcal{H}_Q . Let us call this subspace H_a , and its complementary subspace H_a^\perp .

If $|\psi_1\rangle$ and $|\psi_2\rangle$ are product states, it follows trivially from their linear independence that H_a can be spanned by a pair of orthogonal product states. Its complementary subspace H_a^\perp must also be spanned by a pair of orthogonal product states.

If $|\psi_1\rangle$ and $|\psi_2\rangle$ are entangled states, exactly the same is true—both H_a and H_a^\perp must be spanned by a pair of product states. This is easy to see directly—writing the states in the general forms $|\psi_1\rangle = a|00\rangle + b|11\rangle$ and $|\psi_2\rangle = c|01\rangle + d|10\rangle$, where a, b, c and d are nonzero complex numbers; the states satisfying $|\phi_A\rangle|\phi_B\rangle = |\psi_1\rangle + \sqrt{\frac{ab}{cd}}|\psi_2\rangle$ are the (unnormalized) product states spanning H_a . H_a^\perp is spanned by $\{|\phi_A\rangle|\phi_B^\perp\rangle, |\phi_A^\perp\rangle|\phi_B\rangle\}$.

$|\psi_3\rangle$ is linearly independent of $|\psi_1\rangle$ and $|\psi_2\rangle$, so it has at least some support on H_a^\perp . It therefore has at least some support on one of the two product states spanning H_a^\perp . Let this product state be $|\phi\rangle$. In line with lemma 1, $|\phi\rangle$ is orthogonal to $|\psi_1\rangle$ and $|\psi_2\rangle$, but nonorthogonal to $|\psi_3\rangle$, and therefore $|\psi_3\rangle$ is conclusively locally identifiable. \square

By symmetry, if all the states in \mathcal{S} are entangled, or if they are all product states, then they are all conclusively locally identifiable and \mathcal{S} is conclusively locally distinguishable. If two of them are product states, it is again simple to show they are all conclusively locally identifiable (a consequence of the fact that every set of three orthogonal $2 \otimes 2$ states two of which are product states is perfectly locally distinguishable [13]). But an exception occurs when two of the states are entangled: only can the product state be conclusively locally identified. For example, the set of states:

$$\begin{aligned} |\psi_1\rangle &= \alpha_1|0\rangle_A|0\rangle_B + \alpha_2|1\rangle_A|1\rangle_B, \\ |\psi_2\rangle &= \beta_1|0\rangle_A|0\rangle_B + \beta_2|1\rangle_A|1\rangle_B, \\ |\psi_3\rangle &= |0\rangle_A|1\rangle_B \end{aligned} \tag{3}$$

is not conclusively distinguishable. $|\psi_3\rangle$ is conclusively locally identifiable, but neither $|\psi_1\rangle$ nor $|\psi_2\rangle$ can satisfy the necessary condition for conclusive local identifiability established by lemma 1. This asymmetric possibility is unique to triplets of qubit states, but at least one state can always be identified.

Lemma 4 (many qubits). *Let a multipartite quantum system Q be prepared in one of a set of three pure, linearly independent multipartite quantum states $\mathcal{S} = \{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\}$. Let $\mathcal{H}_Q = \bigotimes_{i=1}^{n>2} \mathcal{H}_2$.*

There exists $|\psi_x\rangle \in \mathcal{S}$ such $|\psi_x\rangle$ is conclusively locally identifiable.

Proof. We begin by considering the Hilbert space of the system with Alice and Bob’s subspaces combined into one four-dimensional subspace \mathcal{H}_{AB} . We can thus write the states:

$$\begin{aligned} |\psi_1\rangle &= \sum_{i,j,\dots=1}^2 a_{ij\dots} |\eta_{ij\dots}\rangle_{AB} |ij\dots\rangle_{CD\dots}, \\ |\psi_2\rangle &= \sum_{i,j,\dots=1}^2 b_{ij\dots} |\nu_{ij\dots}\rangle_{AB} |ij\dots\rangle_{CD\dots}, \\ |\psi_3\rangle &= \sum_{i,j,\dots=1}^2 c_{ij\dots} |\mu_{ij\dots}\rangle_{AB} |ij\dots\rangle_{CD\dots}. \end{aligned} \tag{4}$$

There are $n - 2$ indices i, j, \dots . The states $\{|ij\dots\rangle_{CD\dots}\}$ form an arbitrary canonical basis for the $\bigotimes_{i=1}^{n-2} \mathcal{H}_2$ Hilbert space shared by Carol, Douglas *et al.* The complex coefficients $a_{ij\dots}$, $b_{ij\dots}$ and $c_{ij\dots}$ satisfy normalization constraints.

From lemma 2, we know a state $|\phi\rangle$ exists that is unentangled under the $\mathcal{H}_{AB} \otimes^{n-2} \mathcal{H}_2$ partition and which is orthogonal to $|\psi_1\rangle$ and $|\psi_2\rangle$ but nonorthogonal to $|\psi_3\rangle$. Let us write this state $|\phi\rangle = |\theta\rangle_{AB} \otimes |\omega\rangle_{CD\dots}$. Our choice of canonical basis $\{|ij\dots\rangle_{CD\dots}\}$ for equations (4) was arbitrary, so we can specify retroactively that $|\omega\rangle_{CD\dots} = |00\dots 0\rangle_{CD\dots}$. Then we know that $|\phi\rangle$ satisfies the following equations:

$$\begin{aligned}\langle\psi_1|\phi\rangle &= a_{ij\dots}^* \langle\eta_{ij\dots}|\theta\rangle = 0, \\ \langle\psi_2|\phi\rangle &= b_{ij\dots}^* \langle v_{ij\dots}|\theta\rangle = 0, \\ \langle\psi_3|\phi\rangle &= c_{ij\dots}^* \langle\mu_{ij\dots}|\theta\rangle \neq 0.\end{aligned}\tag{5}$$

Clearly this can be true only if $|\mu_{ij\dots}\rangle_{AB}$ is linearly independent from both $|\eta_{ij\dots}\rangle_{AB}$ and $|v_{ij\dots}\rangle_{AB}$.

$|\eta_{ij\dots}\rangle_{AB}$ and $|v_{ij\dots}\rangle_{AB}$ are either linearly independent of one another, or they are identical. If they are identical, we can trivially find a candidate for $|\theta\rangle_{AB}$ that is a product state in $\mathcal{H}_A \otimes \mathcal{H}_B$, and $|\psi_3\rangle$ is conclusively identifiable state by lemma 1. If they are not identical, then $\{|\eta_{ij\dots}\rangle_{AB}, |v_{ij\dots}\rangle_{AB}, |\mu_{ij\dots}\rangle_{AB}\}$ is a set of three pure linearly independent states, and from lemma 3 there is some product state $|\xi\rangle$ that is nonorthogonal to exactly one of them. In this case, the state $|\xi\rangle_{AB} \otimes |\omega\rangle_{CD\dots}$ is a completely unentangled state in \mathcal{H}_Q satisfying lemma 1 for one of the three states in \mathcal{S} (though not necessarily $|\psi_3\rangle$!). Therefore there is some state in \mathcal{S} that is conclusively locally identifiable. \square

This is the third and final step toward our proof of theorem 1. In all three possible cases, all triplets of pure linearly independent quantum states have been shown to contain a conclusively identifiable state.

If a set of states contains an identifiable member then complete ‘which state’ information is potentially locally discoverable. Otherwise it is necessarily hidden from local observation. In spite of the known links between entanglement and conclusive identifiability, we have shown that any three states can always be locally induced to reveal this information with some probability, no matter how entangled. Furthermore, unless the triplet is a very specific set of multi-qubit states, it is conclusively locally distinguishable and all possible states can be unambiguously identified. An open question is regarding finding optimal local protocols, which would allow a quantitative comparison of the local and global situations.

Acknowledgments

We would like to thank Runyao Duan, Aidan Roy, Anirban Roy, Barry Sanders and Andrew Scott for useful discussions. JW acknowledges support from the Alberta Ingenuity Fund and the Pacific Institute for the Mathematical Sciences, and thanks DIRO at the Université de Montréal for their hospitality with assistance from the Canadian Institute for Advanced Research. SB acknowledges support from Alberta’s Informatics Circle of Research Excellence (iCORE), the Canadian Network of Centres of Excellence for the Mathematics of Information Technology and Complex Systems (MITACS), the Canadian Institute for Advanced Research, General Dynamics Canada, and the Natural Science and Engineering Research Council of Canada (NSERC).

References

- [1] Bennett C H *et al* 1993 *Phys. Rev. Lett.* **70** 1895
Bennett C H and Wiesner S J 1992 *Phys. Rev. Lett.* **69** 20, 2881
- [2] Shor P W 1997 *SIAM J. Sci. Statist. Comput.* **26** 1484

- [3] Bennett C H, DiVincenzo D P, Fuchs C A, Mor T, Rains E, Shor P W, Smolin J A and Wootters W K 1999 *Phys. Rev. A* **59** 1070 (arXiv:quant-ph/9804053)
- [4] Walgate J, Short A J, Hardy L and Vedral V 2000 *Phys. Rev. Lett.* **85** 4972 (arXiv:quant-ph/0007098)
- [5] Helstrom C W 1976 *Quantum Detection and Estimation Theory* (New York: Academic)
- [6] Ivanovic I D 1987 *Phys. Lett. A* **123** 257
Dieks D 1988 *Phys. Lett. A* **126** 303
Peres A 1988 *Phys. Lett. A* **128** 19
- [7] Chefles A 1998 *Phys. Lett. A* **239** 339
- [8] Virmani S, Sacchi M F, Plenio M B and Markham D 2001 *Phys. Lett. A* **288** 62 (arXiv:quant-ph/0102073)
- [9] Ji Z, Cao H and Ying M 2005 *Phys. Rev. A* **71** 032323 (arXiv:quant-ph/0407120)
- [10] Ghosh S, Kar G, Roy A, Sen(De) A and Sen U 2001 *Phys. Rev. Lett.* **87** 277902 (arXiv:quant-ph/0106148)
- [11] Horodecki M, Sen(De) A, Sen U and Horodecki K 2003 *Phys. Rev. Lett.* **90** 047902 (arXiv:quant-ph/0301106)
- [12] Chefles A 2004 *Phys. Rev. A* **69** 050307 (arXiv:quant-ph/0302066)
- [13] Walgate J and Hardy L 2002 *Phys. Rev. Lett.* **89** 147901 (arXiv:quant-ph/0202034)
- [14] Duan R, Feng Y, Ji Z and Ying M 2006 (arXiv:quant-ph/0612034)
- [15] Walgate J and Scott A 2008 *J. Phys. A: Math. Theor.* **41** 375305 (arXiv:0709.4238)